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ABSTRACT

We study the routing problem in massively dense static ad-hoc networks as the node density increases. We use a fluid approximation in which the graph providing the available routes becomes so dense that it can be approximated by a continuous area which inherits from the original problem the cost structure: a cost density is defined at each point on the limit plain; it is a function of the location and the congestion at that point. We solve numerically the routing problem for the case where the cost density is linear with respect to congestion and we obtain a result of convergence via Finite Elements Method.

Categories and Subject Descriptors

C.2.1 [**Network Architecture and Design**]: Wireless Communication; G.1.8 [**Partial Differential Equations**]: Finite element methods

General Terms

Design, Theory, Algorithms

Keywords

Wireless Ad-Hoc Networks, Wireless Sensor Networks, Wardrop Equilibrium, Finite Element Methods

1. INTRODUCTION

We analyse in this paper the routing problem in massively dense static ad-hoc networks. Our basic approach is to design traffic dependent protocols that send each packet along a path that has smallest delay (rather than route packets so as to minimize a global weighted average cost). This criterion goes back to an early paper by Gupta and Kumar [12] who show that by doing so, resequencing delays (that are undesirable in real time traffic and that are very harmful in data transfers using the TCP protocol) are minimized. A recent line of research has been to study such protocols in massively dense static ad-hoc networks. These are characterized by the property that each node has many other nodes in its transmission range. We are interested here in the recent fluid limit approach in which the nodes are modelled as a continuum, and where the discrete graph describing the links and their costs is replaced by a cost density (which depends on the traffic intensity) over the plain. The reason to use such fluid limit approximations is that whereas the complexity of finding optimal routes grows with the number n of nodes, the fluid limit does not depend on n and hence the complexity of finding optimal routes in the fluid approximation does not grow with the number of nodes.

Various physics-inspired paradigms have been used for the study of large ad-hoc networks. Starting from the pioneering work by Jacquet in [17] in that area, a number of research groups have worked on massively dense ad-hoc networks using tools from geometrical optics [17][1]. A second approach was developed based on electrostatics (see e.g. [21, 20, 11] and the survey [22] and references therein).

The physics-inspired paradigms allow the authors to minimize various metrics related to the routing. In contrast, Hyytia and Virtamo propose in [15] an approach based on load balancing arguing that if shortest path (or cost minimization) arguments were used then some parts of the network would carry more traffic than others and may use more energy than others. This would result in a shorter life time of the network since some parts would be out of energy earlier than others and earlier than any part in a load balanced network.

The development of the original theory of routing in massively dense networks among the community of ad-hoc networks has emerged in a complete independent way of the existing related theory which had been developed within the community of road traffic engineers. Indeed, the continuum equilibrium approach had already been introduced on 1952 by Wardrop [23] and by Beckmann [3] and is still an active research area among that community, see [6, 7, 14, 16, 24] and references therein.

We begin this paper by summarizing relevant background and theoretical methodology from [1, 4]. The main contri-

[1]We note that this approach is restricted to costs that do not depend on the congestion.

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bution of this paper is then to propose numerical approaches to approximate the theoretical solutions.

The structure of this paper is as follows: in section 2 we present possible models for cost density. In section 3 we specify the model considered with the linear congestion cost. In section 4 we give some examples which we solve via Finite Element Method. In section 5 we present the conclusions of this work.

2. COST MODELS

2.1 Costs related to the quantity of nodes

Many models have been proposed in the literature that show how the transport capacity scales with the number of nodes n or with their density λ. Assume that we use a protocol that provides a transport capacity of the order of $F(\lambda)$ at some region in which the density of nodes is λ.

A typical cost (used at [20]) at a neighbourhood of x is the density of nodes required there to carry a given flow. Assuming that a flow $\phi(\mathrm{x})$ is assigned through a neighbourhood of x, the cost is taken to be

$$c(\mathrm{x}, \phi(\mathrm{x})) = F^{-1}(\phi(\mathrm{x})). \tag{1}$$

Examples for F:

- Using a network theoretic approach based on multi-hop communication, in the paper of Gupta and Kumar [13] the authors show that the throughput of the system that can be transported by the network when the nodes are optimally located is $\Omega(\sqrt{\lambda})$ and when the nodes are randomly located this throughput becomes $\Omega(\frac{\sqrt{\lambda}}{\sqrt{\log \lambda}})$.

  Using percolation theory, the authors of [8] have shown that in the randomly located set the same $\Omega(\sqrt{\lambda})$ can be achieved.

- Baccelli, Blaszczyszyn and Mühlethaler introduce in [2] an access scheme, MSR (Multi-hop Spatial Reuse Aloha), reaching the Gupta and Kumar bound $\Omega(\sqrt{\lambda})$ which does not require prior knowledge of the node density.

- A protocol introduced by Grossglauser and Tse [10] has a capacity that scales as $\Omega(\lambda)$. However, it does not fall directly within the the class of massively dense static ad-hoc networks because it relies on mobility and on relaying for handling disconnectivity.

2.2 Congestion independent routing

A metric often used in the Internet for determining routing is the number of hops, which routing protocols try to minimize. The number of hops is proportional to the expected delay along the path in the context of ad-hoc networks, in case that the queuing delay is negligible with respect to the transmission delay over each hop. This criterion is insensitive to interference or congestion. We assume that the network is sufficiently dense so that the number of hops does not depend on the density of nodes. It depends only of the transmission range. We describe various cost criteria that can be formulated.

- If the range were constant then the cost density $c(\mathrm{x})$ is constant so that the cost of a path is its length in meters. The routing then follows a shortest path selection.

- Let's assume that the range $R(\mathrm{x})$ depends on local radio conditions at a point x (for example, it is influenced by weather conditions) but not on interference. The latter is justified when dedicated orthogonal channels (e.g. in time or frequency) can be allocated to traffic flows that would otherwise interfere with each other. Then determining the routing becomes a path cost minimization problem. We further assume, as in Gupta and Kumar, that the range is scaled to go to 0 as the total density λ of nodes grows to infinity. More precisely, let's consider a scaling of the range such that the following limit exists:

  $$r(\mathrm{x}) := \lim_{\lambda \to \infty} \frac{R(\mathrm{x})}{F(\lambda)}$$

  where $\lim_{\lambda \to \infty} F(\lambda) = 0$. Then in the dense limit, the density of nodes that participate in forwarding packets along a path is $1/r(\mathrm{x})$ and the path cost is the integral of this density along the path.

- The influence of varying radio conditions on the range can be eliminated using power control that can equalize the hop distance.

2.3 Costs related to energy consumption

In the absence of capacity constraints, the cost can represent energy consumption. In a general multi-hop ad-hoc network the hop distance can be optimized so as to minimize the energy consumption. Even within a single cell of 802.11 IEEE wireless LAN one can improve the energy consumption by using multiple hops, as it has been shown not to be efficient in terms of energy consumption to use a single hop [18].

Alternatively, the cost can take into account the scaling of the nodes. As an example, assuming random deployment of nodes, where each nodes has traffic to send to another randomly selected node, the capacity (in bits per Joule) has then the form $f(\lambda) = \Omega\left((\lambda/\log\lambda)^{(q-1)/2}\right)$ where q is the path-loss, see [19]. The cost is then obtained using (1).

3. MODEL SPECIFICATION

We consider the routing problem in a massively dense static ad-hoc network. A domain Ω of the plane (x, y) is densely covered by potential routers. Messages have to flow from a region $\mathcal{S}$ of the boundary Γ of Ω to a disjoint region $\mathcal{R}$ of Γ. The intensity $\sigma(x, y)$ of message generation on $\mathcal{S}$ is given, while the intensity $\rho(x, y)$ of signal sink on $\mathcal{R}$ is unknown. It is only assumed that these are consistent: the total flow of messages emitted and received are equal. On the rest $\mathcal{T}$ of the boundary of Ω, no message should enter nor leave Ω.

The congestion cost per packet transmitted (say in terms of delays, or energy use) at each point in Ω is a function $c(x, y, \phi)$ of the point and of the intensity ϕ of the flow of messages through that point.

We wish to investigate the optimal routing policy and its relationship with a Wardrop kind of equilibrium.

We follow the model specified in [4] and for the sake of completeness we give here its specification.

We shall use the notation $\mathrm{x} = (x, y)$ to denote a point of $\mathbb{R}^2$. Let Ω be an open domain[2] of $\mathbb{R}^2$ with a smooth

[2]The exact condition is a domain with Lipschitz boundary.

boundary Γ, Ω being at every point of Γ on a single side of Γ, so that an exterior normal to Ω, say $\mathrm{n}(\mathrm{x})$ is well defined and smooth on Γ.

Let the flow of messages be a vector field $f : \Omega \to \mathbb{R}^2$, and $\phi(\mathrm{x}) = \|f(\mathrm{x})\|$ be its intensity. The flux of messages through $\mathcal{S}$ is given as a $\mathcal{C}^1$ function $\sigma(\cdot) : \mathcal{S} \to \mathbb{R}_+$. The consistency assumption reads

$$\int_{\mathcal{R}} \rho(\mathrm{x})\ \mathrm{d}s = \int_{\mathcal{S}} \sigma(\mathrm{x})\ \mathrm{d}s\,. \tag{2}$$

Let $\mathcal{Q} = \mathcal{S} \cup \mathcal{T}$ and extend the function σ to the whole of $\mathcal{Q}$ by $\sigma(\mathrm{x}) = 0$ on $\mathcal{T}$. The boundary conditions will then be

$$\forall \mathrm{x} \in \mathcal{Q} \quad \langle f(\mathrm{x}), \mathrm{n}(\mathrm{x})\rangle = -\sigma(\mathrm{x}). \tag{3}$$

There is no source nor sink of messages in Ω, which as in fluid models, implies the constraint

$$\forall \mathrm{x} \in \Omega \quad \mathrm{div} f(\mathrm{x}) = 0\,. \tag{4}$$

It follows that

$$\int_{\Gamma} \langle f(\mathrm{x}), \mathrm{n}(\mathrm{x})\rangle\, \mathrm{d}s = 0\,,$$

which suffices to insure the consistency condition (2).

The congestion cost per packet c is supposed to be a strictly positive $\mathcal{C}^1$ function $c(\mathrm{x}, \phi) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+\backslash\{0\}$, increasing and convex in ϕ for each x. The total cost of congestion will be taken as

$$G(f(\cdot)) = \int_{\Omega} c(\mathrm{x}, \|f(\mathrm{x})\|)\|f(\mathrm{x})\|\ \mathrm{d}\mathrm{x}\,. \tag{5}$$

The path followed by a packet is specified by its direction of travel $e_\theta = (\cos\theta,\ \sin\theta)$ along its path, according to $\dot{\mathrm{x}} = e_\theta$. The cost incurred by one packet travelling from $\mathrm{x}_0 \in \mathcal{S}$ at time t_0 to $\mathrm{x}_1 \in \mathcal{R}$ reached at time t_1 is

$$J(e_\theta(\cdot)) = \int_{\Gamma} c(\mathrm{x}, \|f(\mathrm{x})\|)\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} \tag{6}$$

$$= \int_{t_0}^{t_1} c(\mathrm{x}(t), \|f(\mathrm{x}(t))\|)\ \mathrm{d}t\,. \tag{7}$$

where Γ is the path such that $(x(t_0), y(t_0)) = \mathrm{x}_0$ and $(x(t_1), y(t_1)) = \mathrm{x}_1$ following at each time t the path given by $e_\theta(\cdot)$.

Notice that this "time" t may be a fictitious time, related to physical time, say τ, by $\mathrm{d}\tau = c\ \mathrm{d}t$ for instance. Then c is the inverse of a speed of travel, a delay due to congestion, and J is the time taken by the message to go from source to destination.

The development of the general problem has been studied in [4].

Quadratic congestion total cost

In this work we are going to focus on the quadratic congestion total cost that has been studied by Tassiulas and Toumpis [20] considering the cost related to the quantity of nodes deduced from Gupta and Kumar [13]. We assume that the cost of congestion is linear, i.e.

$$c(\mathrm{x}, \phi) = \frac{1}{2}c(\mathrm{x})\phi,$$

which implies that the function that we must minimize for the global optimization problem is

$$G(\mathrm{x}, \phi) = \frac{1}{2}\int_{\Omega} c(x)\phi^2\ \mathrm{d}x.$$

The necessary optimality condition for the global optimization problem gives us that if we find p such that

$$\forall \mathrm{x} \in \Omega \quad \mathrm{div}\left(\frac{1}{c(\mathrm{x})}\nabla p(\mathrm{x})\right) = 0, \tag{8a}$$

$$\forall \mathrm{x} \in \mathcal{Q} \quad \frac{\partial p}{\partial n}(\mathrm{x}) = c(\mathrm{x})\sigma(\mathrm{x}), \tag{8b}$$

$$\forall \mathrm{x} \in \mathcal{R} \quad p(\mathrm{x}) = 0. \tag{8c}$$

Then p is optimal.

From [4] we deduce that the optimal flow is given in the linear case by

$$f^*(\mathrm{x}) = \frac{\nabla p(\mathrm{x})}{c(\mathrm{x})}.$$

The authors of [4] defined a continuum Wardrop equilibrium where each single message seek its optimal solution to follow the path that minimizes the integral line of the cost function across the path, assuming that its lone deviation from that scheme would have no effect on the overall congestion map.

In that setting the Wardrop equilibrium can be obtained by solving the globally optimal problem in which the total cost density is replaced by $\int_0^\phi c(\mathrm{x}, \phi)\,\mathrm{d}\phi$. That implies that the global optimal solution and the Wardrop equilibrium coincide in the domain $\{\mathrm{x} \in \Omega; f^*(\mathrm{x}) \neq 0$.

In order to find a numerical solution to our problem we consider the Finite Element Method, largely used in numerical modelling of physical systems in disciplines such as for example, Electromagnetism and Fluid Dynamics.

The general variational problem is to seek $u \in V$ such that

$$(VP)\begin{cases} a(u, v) = l(v) \\ \forall\, v \in V. \end{cases}$$

We seek the weak formulation of the set of equations (8). If we multiply equation (8a) by a generic function $\psi \in H^1_{\mathcal{R}}(\Omega)$ where we consider the closed subspace

$$H^1_{\mathcal{R}}(\Omega) = \{v \in H^1(\Omega);\, v(\mathrm{x}) = 0 \quad \forall\, \mathrm{x} \in \mathcal{R}\}$$

and integrate over the domain Ω, after a small manipulation using equations (8b) and (8c), we obtain that equation (8a) is equivalent to

$$\int_{\Omega} \frac{1}{c}\nabla p \cdot \nabla\psi\, \mathrm{d}x = \int_{\mathcal{Q}} \sigma \cdot \psi\, \mathrm{d}x.$$

The problem defined by the set of equations (8) is equivalent to the problem of seeking $p \in H^1_{\mathcal{R}}(\Omega)$ such that

$$\int_{\Omega} \frac{1}{c}\nabla p \cdot \nabla\psi\, \mathrm{d}x = \int_{\mathcal{Q}} \sigma \cdot \psi\, \mathrm{d}x \quad \forall \psi \in H^1_{\mathcal{R}}(\Omega).$$

In that sense, if we consider the functions $a(\cdot,\cdot)$ and $l(\cdot)$ defined as

$$a(u, v) = \int_{\Omega} \frac{1}{c}\nabla u \cdot \nabla v\, \mathrm{d}x\,,$$

and

$$l(v) = \int_{\mathcal{Q}} \sigma \cdot v\, \mathrm{d}x.$$

in the space $V = H^1_{\mathcal{R}}(\Omega)$, we have set our problem as a variational problem, where the solution will be the function p.

In our case, the bilinear function $a(\cdot,\cdot)$ is V-elliptic, symmetric and continuous in $H^1(\Omega)$ and the linear function $l(\cdot)$

is bounded. Then we can use Lions-Lax-Milgram Theorem and conclude that the solution exists and is unique.

This theorem gives us not only the existence and uniqueness of the solution but also gives us information about the stability of the solution when the data changes saying that the solution depends continuously on the data.

3.1 Incentives to use the Finite Element Method

The idea of the Finite Element Method is to discretize the problem (VP) when the dimension of the space is infinite. This is interesting in our case as we are looking for a solution within the space of functions $H^1(\Omega)$. From this approach we obtain a linear system for which there are many standard methods to solve it.

An intern discretization of the variational problem is to take V_h as a vector subspace of finite dimension ($V_h \subseteq V$), where $h > 0$ is a discretization parameter such that when $h \to 0$, the dimension of V_h goes to infinity.

As $V_h \subseteq V$, $a(\cdot,\cdot)$, $l(\cdot)$ are well defined in V_h, then the discretized problem becomes to seek $u_h \in V_h$ such that

$$(EV_h) \begin{cases} a(u_h, v_h) = l(v_h) \\ \forall v_h \in V_h, \quad \forall u_h \in V_h. \end{cases}$$

Due to Lions-Lax-Milgram we have a solution that depends continuously on the data.

Let $\{\phi_1, \ldots, \phi_{N_h}\}$ be a base of V_h, then we can write for every $u_h \in V_h$,

$$u_h = \sum_{j=1}^{N_h} \alpha_j \phi_j,$$

where $\alpha = (\alpha_1, \ldots, \alpha_{N_h}) \in \mathbb{R}^{N_h}$ is unique for each $u_h \in V_h$. Then the equation of the (EV_h) problem for the base of V_h becomes

$$a\left(\sum_{j=1}^{N_h} \alpha_j \phi_j, \phi_i\right) = l(\phi_i) \quad \forall i = 1, \ldots, N_h,$$

Then from the bilinearity and symmetry of $a(\cdot,\cdot)$, we obtain

$$\sum_{j=1}^{N_h} a(\phi_i, \phi_j)\alpha_j = l(\phi_i) \quad i = 1, \ldots, N_h, \quad \alpha \in \mathbb{R}^{N_h},$$

which is equivalent to the linear system

$$A\alpha = b,$$

where $\alpha \in \mathbb{R}^{N_h}$, $A_{ij} = a(\phi_j, \phi_i)$, and $b_i = l(\phi_i)$. In order to solve this linear system, we can use standard methods.

3.2 Description of the Finite Element Method

In this part we are going to give some definitions to understand the finite element method.

For every $k \geq 0$, we denote $\mathbb{P}_k$ to the *space of polynomial functions* from $\mathbb{R}^2$ to $\mathbb{R}$ of degree less or equal to k.

Definition.- [2-simplex] Consider 3 vertices

$$\begin{aligned} a_1 &= (a_{11}, a_{21}) \\ a_2 &= (a_{12}, a_{22}) \\ a_3 &= (a_{13}, a_{23}) \end{aligned}$$

in $\mathbb{R}^2$, not aligned (we call them non-degenerated vertices), then a 2-simplex T of vertices $\{a_j\,;\, 1 \leq j \leq 3\}$ is the convex hull of those vertices.

As the vertices are not in the same line, every point $\mathrm{x} \in \mathbb{R}^2$ can be written as a linear combination of those vertices. We denote $\{\lambda_j(\mathrm{x})\,;\, 1 \leq j \leq 3\}$ the *barycentric coordinates* of the point x with respect to the vertices $\{a_j,\, 1 \leq j \leq 3\}$.

Then, we can characterize the 2-simplex T of vertices $\{a_j, 1 \leq j \leq 3\}$ by

$$T = \left\{ x = \sum_{i=1}^{3} \lambda_j(x)\, a_j \,;\, 0 \leq \lambda_j(x) \leq 1\,,\, \forall\, 1 \leq j \leq 3 \right\}.$$

For the error analysis it is useful to consider the following geometric parameters:

$$\begin{aligned} h_T &= \text{diameter of } T \\ &= \text{Length of the greatest side,} \\ \rho_T &= \text{roundness of } T \\ &= \text{Diameter of the greatest ball included in } T. \end{aligned}$$

We define the 2-simplex of reference $\hat{T}$ as the 2-simplex that has as vertices $\hat{a}_1 = (1,0), \hat{a}_2 = (0,1)$ and $\hat{a}_3 = (0,0)$.

If we consider T as a non-degenerated 2-simplex of vertices $\{a_j, 1 \leq j \leq 3\}$, then there exists a unique invertible matrix $B_T \in \mathbb{R}^{2,2}$ and a unique vector $b_T \in \mathbb{R}^2$ such that

$$\forall\, 1 \leq j \leq 3\,,\, a_j = B_T \hat{a}_j + b_T.$$

We denote $\mathcal{F}_T$ this affine transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

3.3 Construction of the finite elements

Let us consider:

- A compact set $T \subseteq \mathbb{R}^2$, connected and with non-empty interior.
- A finite set $\Sigma = \{a_j\}_{j=1}^N$ of N distinct points of T.
- A vectorial space P of finite dimension and made of functions from T to the set of real numbers.

Definition.- [P-unisolvent] The set Σ is called P-unisolvent, if and only if, given N scalars α_j, $1 \leq j \leq N$, there exists a function p on the space P and only one such that

$$p(a_j) = \alpha_j, \quad 1 \leq j \leq N.$$

When the set Σ is P-unisolvent, the triplet (T, P, Σ) is called *finite element of Lagrange*.

Given these definitions we have that for every function v defined over T to real values, there exists a function $p \in P$ and only one, that interpolate v over Σ, i.e. it satisfies

$$p(a_j) = v(a_j),\ 1 \leq j \leq N.$$

Definition.- Given a finite element of Lagrange (T, P, Σ), we call *functions of base* to the N functions p_i, $1 \leq i \leq N$, such that

$$p_i(a_j) = \delta_{ij}, \quad 1 \leq j \leq N.$$

We call operator of P-interpolation of Lagrange over Σ to the operator that to any function v defined over T it gives

$$\Pi v = \sum_{i=1}^{N} v(a_i) p_i,$$

and Πv is called the P-interpolate of Lagrange of v over Σ.

Theorem.-[9],Theorem 4.5.) Let $(\hat{T}, \hat{P}, \hat{\Sigma} = \{\hat{\varphi}_i; 1 \leq i \leq M\})$ be a finite element of Lagrange on $\mathbb{R}^2$ and let F be

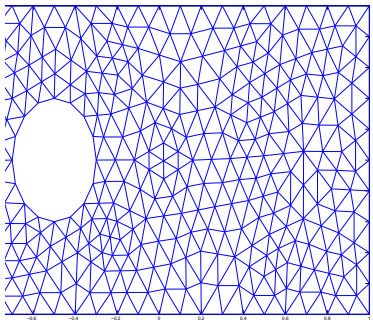

Figure 1: Triangulation of the domain $[-1,1]\times[-0.5,0.5]\backslash D((-0.5,0),0.2)$.

a bicontinuous bijective function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then the triplet $(T,P,\Sigma=\{\varphi_i;1\leq i\leq M\})$ defined by

$$T = F(\hat{T}) \tag{9a}$$

$$P = \{\hat{p}\circ F^{-1};\hat{p}\in\hat{P}\} \tag{9b}$$

$$\mathrm{dom}(\Sigma) = \{v=\hat{v}\circ F^{-1};\hat{v}\in\mathrm{dom}(\hat{\Sigma})\} \tag{9c}$$

$$\forall v\in\mathrm{dom}(\Sigma),\varphi_i(v)=\hat{\varphi}_i(\hat{v}),1\leq i\leq M, \tag{9d}$$

is also a finite element of Lagrange.

Definition.- Two finite elements of Lagrange $(\hat{T},\hat{P},\hat{\Sigma}=\{\hat{\varphi}_i;1\leq i\leq M\})$ and $(T,P,\Sigma=\{\varphi_i;1\leq i\leq M\})$ are equivalents if there exists a bicontinuous bijective function F from $\hat{T}$ to T such that (T,P,Σ) satisfy (9). If F is an affine transformation, we say that they are *affine-equivalents*.

From the elements described above we realize that we can construct from the 2-simplex of reference $\hat{T}$ all the 2-simplex that we want and that these 2-simplex will be affine-equivalents to the 2-simplex of reference.

Suppose the domain Ω has a polygonal boundary Γ, then we cover the closure of Ω by a triangulation, i.e.

$$\bar{\Omega} = \bigcup_{T\in\mathcal{T}} T$$

and each T is a closed triangle where

- Every $T\in\mathcal{T}$ is a triangle.
- The interior of two different triangles are disjoint.
- Every face of a triangle is either the face of another triangle (in which case, they are called adjacent) or a part of the boundary.

As an example, in Figure 1 we have a rectangular domain of boundary $[-1,1]\times[-1,1]$ minus the disk of centre $(-0.5,0)$ and radius 0.2, and triangulation approximation defined on this domain.

For convention, $\mathcal{T}_h$ denotes a triangulation of $\bar{\Omega}$ such that

$$h = \max_{T\in\mathcal{T}_h} h_T\,,$$

where h_T is the diameter of the polygon T.

Suppose that for every polygon T of $\mathcal{T}_h$, there is associated a finite element of Lagrange (T,P_T,Σ) such that

$$P_T\subseteq H^1(T)\,,$$

and we define the finite dimensional spaces

$$X_h = \{v\in\mathcal{C}^0(\bar{\Omega})\,;\,\forall T\in\mathcal{T}_h,\,v|_T\in P_T\} \tag{10}$$

$$X_{0h} = \{v\in X_h\,;\,v|_{\mathcal{R}}=0\}. \tag{11}$$

If (T,P,Σ) is a finite element of Lagrange, to any function v defined over T, we associate the function Πv that P-interpolates v over T. The idea is to study an upperbound for the interpolation error $v-\Pi v$ with the norm $H^1(T)$.

Let T be a compact of $\mathbb{R}^2$, connected and with non-empty interior. For simplicity we denote $H^m(T)$ at the Sobolev space $H^m(\mathring{T})$ where $\mathring{T}$ is the interior of T.

Definition.-

Let T be a polygon on $\mathbb{R}^2$. A finite element (T,P,Σ) is called of class $\mathcal{C}^0$ if the two following conditions are satisfied:

1. $$P\subseteq\mathcal{C}^0(T),$$
2. For every face T' of T, the set $\Sigma'=\Sigma\cap T'$ is P'-unisolvent where $P'=\{p|_{T'};\,p\in P\}$

Definition.-

We call $(\mathcal{T}_h)$ a family of regular triangulations of $\bar{\Omega}$ if the following four conditions are satisfied:

1. All the finite elements (T,P_T,Σ_T) of every triangulation are affine-equivalents to the same finite element of reference $(\hat{T},\hat{P},\hat{\Sigma})$ of class $\mathcal{C}^0$.
2. For every pair $(\hat{T}_1',\hat{T}_2')$ of faces of $\hat{T}$ and for every application $\hat{F}$ affine invertible and from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $\hat{T}_2'=\hat{F}(\hat{T}_1')$, we have
   $$\hat{\sigma}\cap\hat{T}_2' = \hat{F}(\hat{\sigma}\cap\hat{T}_1')$$
   and
   $$\{\hat{p}|_{K_2'};\,\hat{p}\in\hat{P}\} = \{p\circ\hat{F}|_{K_1'};\,p\in\hat{P}\}.$$
3. We have
   $$h = \max_{T\in\mathcal{T}_h} h_T\to 0.$$
4. There exists a constant $\sigma\geq 1$ such that
   $$\forall h,\,\forall T\in\mathcal{T}_h,\,\frac{h_T}{\rho_T}\leq\sigma.$$

Theorem.- Let Ω be an open polytope of $\mathbb{R}^d$, $d\leq 3$. Let $(\mathcal{T}_h)$ be a family of regular triangulations of $\bar{\Omega}$ associated to a finite element of reference $(\hat{T},\hat{P},\hat{\Sigma})$ of class $\mathcal{C}^0$.

We suppose that there exists an integer $k\geq 1$ such that

$$P_k\subseteq\hat{P}\subseteq H^1(\hat{K})$$

Then the finite element method is convergent, i.e. the solution u_h of the problem (VP_h) converges to the solution of (VP) in $H^1(\Omega)$:

$$\lim_{h\to 0}\|u-u_h\|_{1,\Omega}=0.$$

There exists a constant C independent of h such that if the solution belongs to the Sobolev space $H^{k+1}(\Omega)$

$$\|u-u_h\|_{1,\Omega}\leq Ch^k|u|_{k+1,\Omega}.$$

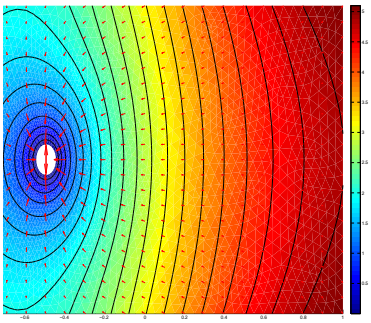

Figure 2: Solution for a wireless sensor network.

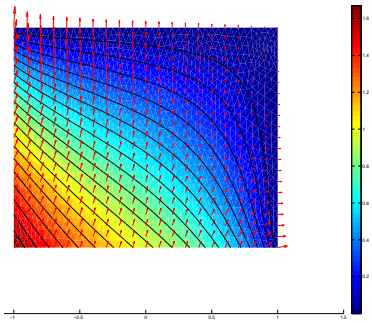

Figure 3: Problem similar to Dafermos' problem.

4. EXAMPLES OF APPLICATIONS

1.- Gathering in Wireless Sensor Networks

Suppose that we want to protect a land for growing crops from an external threat such as forest fire. We deploy uniformly a large quantity of sensor nodes over the land and we want to give a description of the flow according to this setting and a description of the arrival of the new information that is coming from the sensor nodes that are on the boundary of the land. As we are considering an external threat we consider that only the external nodes are going to generate information and the interior nodes are going to serve only as relay nodes.

We suppose that the generation of information can be approximated by $\sigma = 1$ for every sensor node in the boundary.

We consider also that the centre of analysis of information is located inside the domain and for simplicity we suppose that can be modelled as a closed set with non-empty interior.

We suppose that our land can be modelled by the rectangle $[-1, 1] \times [-0.5, 0.5]$ and the centre of analysis of information is located at the point $(-0.5, 0)$. If we suppose that we distribute uniform over the whole network, then is reasonable to suppose if the land doesn't have other environmental problems the cost of the network will be uniform and for simplicity we consider $c = 1$.

After using the modelling explained in the previous section the direction of the flow of information can be described by the red lines in Figure 2.

2.- Dafermos Example

We consider a rectangular domain $[-1, 1]\times[-0.5, 0.5]$. Following the paper of Dafermos [6] we impose that the value of the flow in the boundary is uniform and equal to 1 in the vertical left boundary and in the horizontal lower boundary.

We suppose that we distribution of the nodes inside the domain is uniform, then is reasonable to suppose if the land doesn't have other environmental problems the cost of the network will be uniform and for simplicity we consider $c = 1$.

Then the description of the flow of information is given by the red lines in Figure 3.

3.- Example with obstacles

Once more, for simplicity we consider a rectangular domain but this time for problems in the land we consider that some of the relay nodes can not be put in a specific area. We suppose that the distribution of the nodes is uniform and we suppose that the cost of the network will be uniform $c = 1$.

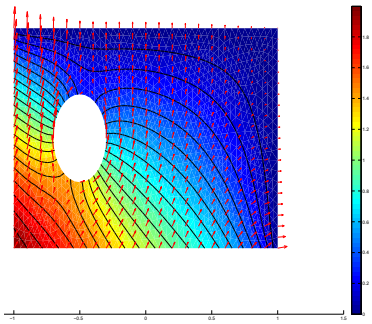

Figure 4: When the domain considered is not simply connected.

We consider that the domain is give by $[-1, 1]\times[-0.5, 0.5]$ and the hole is modelled as a circle with centre in $(-0.5, 0)$ and radius 0.2.

Then the direction of the flow of information is given by the red lines in Figure 4.

5. CONCLUSIONS

In the present work we have used the Finite Element Method in solving the routing problem in massively dense static ad-hoc networks. The node density in these massively dense systems is approximated by a continuous area with costs depending on the location and the congestion of the network.

The problem considered is that messages have to flow from a region $\mathcal{S}$ of the boundary Γ of a domain Ω to a disjoint region $\mathcal{R}$ of Γ. The intensity of message generation on $\mathcal{S}$ is given. In this framework we study the case of linear congestion cost per packet. We mention a result from [4] on existence and uniqueness of the solution and present the stability of this solution with respect to the initial flow.

Numerically we obtain via the Finite Element Method an approximation of the solution and prove a result of convergence and the velocity of convergence of this numerical approximation to the exact solution.

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APPENDIX

Notations

Some of the notation used in this paper are of current use in Mathematics but may not be known for non-mathematicians.

The functions in $L^1_{\text{loc}}(\Omega)$, also called locally integrable functions, are the functions which are integrable on any compact set of its domain of definition Ω .

The functions in $L^2(\Omega)$ are the functions that are square integrable, i.e.

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{R}; \int_\Omega |f(x)|^2 \, dx < +\infty \right\}.$$

Note: The functions in $L^2(\Omega)$, also belong to $L^1_{\text{loc}}(\Omega)$.

The functions in $\mathcal{C}_c(\Omega)$ are the continuous functions with compact support over Ω. The functions in $\mathcal{C}^k(\Omega)$ are the functions k times continuously differentiable over Ω ($k \geq 0$).

The function in $\mathcal{C}^\infty(\Omega) = \bigcap_{k \geq 0} \mathcal{C}^k$ are the functions for which all its derivatives are differentiable over Ω.

The function in $\mathcal{C}^\infty_c = \mathcal{C}^\infty \cap \mathcal{C}_c(\Omega)$ are the functions with compact support over Ω for which all its derivatives are differentiable over Ω.

Given a function $u \in \mathcal{C}^1(\Omega)$ and a function $\phi \in \mathcal{C}^\infty_c$, using the integration by parts formula we obtain

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} \, dx = - \int_\Omega \frac{\partial u}{\partial x_i} \phi \, dx \quad \forall i = 1, 2. \tag{12}$$

There are no boundary terms, since ϕ has compact support in Ω and thus vanishes near the boundary $\partial\Omega$. The left hand side of equation (12) makes sense even if $u \in L^1_{\text{loc}}(\Omega)$. However, the expresion $\frac{\partial u}{\partial x_i}$ on the right hand side doesn't have a meaning. Then in Mathematics people work with a concept called weak partial derivative.

Definition.- Suppose $u, v \in L^1_{\text{loc}}(\Omega)$. We say that v is the weak partial derivative of u, written $v = Du$ provided

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = - \int_\Omega v_i \phi \quad \forall \phi \in \mathcal{C}^\infty_c(\Omega) \quad \forall i = 1, 2.$$

The functions in $H^1(\Omega)$ are the functions that are square integrable and whose weak partial derivative is also square integrable.

$$H^1(\Omega) = \left\{ u \in L^2(\Omega); Du \in L^2(\Omega) \right.$$

For more information about the reason to work on these spaces consult chapter VIII and IX of the book on Functional Analysis [5].